On the absolute summability factors of Fourier series

By LÁSZLÓ LEINDLER in Szeged*)

Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\lambda=\{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1}-\lambda_n\leqq 1$ and $\lambda_1=1$.

The sequence-to-sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n+1}^{n} s_\nu$$

defines the sequence $\{V_n(\lambda)\}$ of generalized de la Vallée Poussin means of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$. The series $\sum a_n$ is said to be summable $|V,\lambda|$, if the series

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda)-V_n(\lambda)|$$

is convergent. Let $\lambda(x)$ $(x\geqq 1)$ be a continuous function linear between n and $n+1$, furthermore $\lambda(n)=\lambda_n$.

Let $f(x)$ be a function integrable in the sense of Lebesgue and periodic with period 2π and let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) = \sum A_n(x). \tag{1}$$

For a fixed of x, we write

$$\varphi(t)=\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$$

and

$$\Phi(t) = \int_0^t |\varphi(u)|\,du.$$

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Let $\{\mu_\nu\}$ be a convex and bounded sequence. CHOW [1] proved that *the series*

$$\sum A_n(z)\mu_n \tag{2}$$

is summable $|C, 1|$[1]) *almost everywhere, if the series* $\sum n^{-1}\mu_n$ *converges.* CHOW [1] proved also that, *if $f(x)$ belongs to the class H, i. e. if $f(x)$ and its conjugate function are both L-integrable, and if $\{\mu_n\}$ is a sequence such that the series*

$$\sum_{n=1}^{\infty} n(\Delta\mu_n)^2, \quad \sum_{n=1}^{\infty} \frac{\mu_n^2}{n} \tag{3}$$

converge, then the series (2) *is summable* $|C, 1|$ *for almost all values of z. If μ_n is convex, it is sufficient to assume the convergence of the second series* (3).

Later PATI [4] proved: *if* $\sum n^{-1}\mu_n < \infty$ *and at a fixed point of x* $\Phi(t) = O(t)$ $(t \to 0)$, *then the series* (2) *is summable* $|C, \alpha|$ *for every* $\alpha > 1$ *at this point* $z = x$.

Recently HSIANG [2] demonstrated: *if at a fixed point of x* $\Phi(t) = (t(\log 1/t)^{-1})$ $(t \to 0)$, *then the series* $\sum A_n(x)/(\log n)^{1+\varepsilon}$ *is summable* $|C, 1|$ *for every* $\varepsilon > 0$.

In this field many other interesting results have been obtained mostly by Indian and Chinese mathematicians.

In the present paper, we are going to give some theorems of $|V, \lambda|$-summability, similar to the cited ones. Since in some of our structural conditions both the magnitude of the factor sequence $\{\mu_n\}$ and the strength of the summability appear we obtain new results even in the classical case of $|C, 1|$-summability.

Theorem 1. *If $\{\mu_n\}$ is a monotone convex sequence and the series* $\sum \mu_n \lambda_n^{-1}$ *converges, then the series* (2) *is summable* $|V, \lambda|$ *almost everywhere.*

Theorem 2. *If $f(x)$ belongs to the class H and if $\{\mu_n\}$ is monotone convex and satisfies the condition* $\sum n\mu_n^2 \lambda_n^{-2} < \infty$, *then the series* (2) *is summable* $|V, \lambda|$ *almost everywhere.*

The following theorems concern the summability at a given point.

Theorem 3. *Let $\mu(x)$ $(x \geqq 0)$ be a function monotone decreasing and satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{\lambda_n} < \infty. \tag{4}$$

If

$$\Phi(t) = O\left(\lambda^{-1}\left(\frac{1}{t}\right)\mu\left(\frac{1}{t}\right)\right) \tag{5}$$

as $t \to +0$, then the series

$$\sum_{n=1}^{\infty} \mu(n) A_n(x) \tag{6}$$

is summable $|V, \lambda|$ *at the point x.*

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[1]) A series $\sum c_n$ is said to be summable $|C, \alpha|$ $(\alpha \geqq 0)$ if $\sum |\sigma_{n+1}^\alpha - \sigma_n^\alpha|$ where σ_n^α is its n-th Cesàro mean of order α converges.

Theorem 4. *If instead of* (4) *the condition*

$$\sum_{n=4}^{\infty} \frac{\mu(n) \log \log n}{\lambda_n} < \infty \tag{7}$$

is fulfilled, then the condition

$$\Phi(t) = O\left(t\left(\log \frac{1}{t}\right)^{-1}\right) \tag{8}$$

also suffices for the $|V, \lambda|$*-summability of the series* (6).

In this theorem the structural condition (8) is independent of the factor sequence and the summability.

In the special case $\lambda_n = n$, i. e. in the case of $|C, 1|$-summability, Theorems 3 and 4 give the following result:

If either $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} < \infty$ and $\Phi(t) = O\left(t\mu\left(\frac{1}{t}\right)\right)$, or $\sum_{n=4}^{\infty} \frac{\mu(n) \log \log n}{n} < \infty$ and (8) are fulfilled, then the series (6) is summable $|C, 1|$.

From this, by Lemma 4, we have as

Corollary. *Let* $\{p_n\}$ *be a non-increasing sequence of positive numbers and let* $P_n = p_0 + p_1 + \ldots + p_n$. *If either* $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} < \infty$ *and* $\Phi(t) = O\left(t\mu\left(\frac{1}{t}\right)\right)$, *or* $\sum_{n=4}^{\infty} \frac{\mu(n) \log \log n}{n} < \infty$ *and* (8) *are fulfilled, then the series*

$$\sum_{n=1}^{\infty} \frac{\mu(n) P_n A_n(x)}{n}$$

is summable $|N, p_n|$. [2])

Finally we prove the following

Theorem 5. *Let* $0 < \alpha \leqq 1$ *and let* $\mu(x)$ $(x \geqq 0)$ *be a function monotone decreasing and satisfying either*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} < \infty \quad \textit{and} \quad \Phi(t) = O\left(t^\alpha \mu\left(\frac{1}{t}\right)\right), \tag{9}$$

or

$$\sum_{n=4}^{\infty} \frac{\mu(n) \log \log n}{n^\alpha} < \infty \quad \textit{and} \quad \Phi(t) = O\left(t\left(\log \frac{1}{t}\right)^{-1}\right), \tag{10}$$

then the series (6) *is summable* $|C, \alpha|$.

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[2]) A series $\sum a_n$ is said to be summable $|N, p_n|$ if $\sum |t_{n+1} - t_n|$, where $t_n = \sum_{k=0}^{n} \frac{p_k s_{n-k}}{P_n}$ is the n-th Nörlund mean, converges.

From the condition (9), it is easy to see the close connection existing between the power of the summability and the magnitude of the factor sequence.

It is clear, too, that in the special case $\mu(x)=(\log x)^{-1-\varepsilon}$ and $\alpha=1$ the second half of Theorem 5 includes the theorem of HSIANG.

It seems worth while to observe that we can derive analogous structural theorems for the conjugate series

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(x).$$

Write

$$\Psi(t)=\int_0^t|f(x+u)-f(x-u)|\,du.$$

Then, for example we have the following:

Theorem 6. *Let $\mu(x)$ $(x\geqq 0)$ be a function monotone decreasing and satisfying either*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\lambda_n}<\infty \quad \text{and} \quad \Psi(t)=O\left(t\mu\left(\frac{1}{t}\right)\right) \qquad (t\to 0),$$

or

$$\sum_{n=4}^{\infty}\frac{\mu(n)\log\log n}{\lambda_n}<\infty \quad \text{and} \quad \Psi(t)=O\left(t\left(\log\frac{1}{t}\right)^{-1}\right) \qquad (t\to 0),$$

then the series

$$\sum_{n=1}^{\infty}\mu(n)B_n(x)$$

is $|V,\lambda|$-summable.

The second half of this theorem in case $\lambda_n=n$ and $\mu(x)=(\log x)^{-1-\varepsilon}$ includes the theorem of HSIANG [2] given for the conjugate series.

§ 1. Lemmas

We require the following lemmas.

Lemma 1. (cf. [1], Lemma 2.) *Let*

$$t_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}kA_k(x).$$

Then

$$\sum_{k=1}^{n}|t_k(x)|=o(n)$$

for almost all values of x.

Lemma 2. (cf [1], Lemma 7.) *If $f(x)$ belongs to the class H, the series $\sum n^{-1}|t_n(x)|^2$ converges for almost all values of x.*

Lemma 3. *If $\{\mu_n\}$ is convex and $\sum n^{-1}\mu_n^2<\infty$, then $\sum n(\Delta\mu_n)^2$ converges.*

This lemma holds by the proof of Theorem 2 of CHOW [1].

Lemma 4. (cf. [3].) *If a series $\sum a_n$ is summable $|C,1|$ and if $\{p_n\}$ is a non-increasing sequence of real and non-negative numbers, then the series $\sum a_n P_n n^{-1}$ is summable $|N,p_n|$.*

Lemma 5. (cf. [2], Lemma 2.) *Denote*

$$C_n(t)=\sum_{k=1}^{n} k\cos kt,$$

then

$$C_n(t)=O(nt^{-1})$$

for $nt\geqq 1$.

§ 2. Proofs of the theorems

Proof of Theorem 1. An easy computation gives that

$$V_{n+1}(\lambda)-V_n(\lambda)=\frac{1}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}[(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n]a_k.$$

Let $V_n(\lambda;z)$ denote the n-th de la Vallèe Poussin mean of the series (2). Then we have that

$$\sum_{n=1}^{\infty}|V_{n+1}(\lambda;z)-V_n(\lambda;z)|=$$

$$=\sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}[(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n]\mu_k A_k(z)\right|.$$

Let $\sum_n'$ be the summation over all n satisfying $\lambda_{n+1}=\lambda_n$; and $\sum_n''$ the summation over all n where $\lambda_{n+1}>\lambda_n$. Then, ABEL's transformation gives that

$$\sum_n'=\sum_n'\frac{1}{\lambda_{n+1}}\left|\sum_{k=n-\lambda_n+2}^{n+1}\frac{\mu_k}{k}kA_k(z)\right|=$$

$$=\sum_n'\frac{1}{\lambda_{n+1}}\left\{\sum_{k=n-\lambda_n+2}^{n}\left|\sum_{\nu=1}^{k}\nu A_\nu(z)\right|\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)-\right.$$

$$\left.-\frac{\mu_{n-\lambda_n+2}}{n-\lambda_n+2}\left|\sum_{\nu=1}^{n-\lambda_n+1}\nu A_\nu(z)\right|+\frac{\mu_{n+1}}{n+1}\left|\sum_{\nu=1}^{n+1}\nu A_\nu(z)\right|\right\}\equiv\sum_n{}_1'+\sum_n{}_2'+\sum_n{}_3'.$$

Since the inside lower indices $n-\lambda_n+2$ in $\sum_n{}_1'$ are strictly increasing, we have

$$\sum_n{}_1'=O(1)\sum_{k=1}^{\infty}k|t_k(z)|\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)\sum_{n=k}^{k+\lambda_k-1}\frac{1}{\lambda_n}=O(1)\sum_{k=1}^{\infty}k|t_k(z)|\,\Delta\left(\frac{\mu_k}{k}\right)\equiv A(z).$$

It is easy to see that

$$\sum_n{}'_2+\sum_n{}'_3 = O(1)\sum_{n=1}^{\infty}\frac{\mu_n}{\lambda_n}|t_n(z)| \equiv B(z).$$

Using Abel's transformation again, by Lemma 1, we get

$$A(z) = O(1)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k|t_k(z)|\right)\Delta^2\left(\frac{\mu_n}{n}\right) = O(1)\sum_{n=1}^{\infty}n^2\Delta^2\left(\frac{\mu_n}{n}\right)$$

for almost all z. Since

$$\Delta^2\left(\frac{\mu_n}{n}\right) \leqq \frac{1}{n}\Delta^2\mu_n+\frac{2}{n^2}\Delta\mu_n+\frac{\mu_n}{n^3},$$

we have

$$A(z) = O(1)\sum_{n=1}^{\infty}n\Delta^2\mu_n+O(1) = O(1)\sum_{n=1}^{\infty}\Delta\mu_n+O(1) = O(1) \tag{2. 1}$$

for almost all z. Similarly, by Lemma 1, it follows that

$$B(z) = O(1)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}|t_k(z)|\right)\Delta\left(\frac{\mu_n}{\lambda_n}\right) = O(1)\sum_{n=1}^{\infty}n\,\Delta\left(\frac{\mu_n}{\lambda_n}\right) = O(1) \tag{2. 2}$$

holds for almost all z. This means that the sum $\sum_n'$ converges almost everywhere.

The estimation of $\sum_n''$ is somewhat more tricky. We obtain, with the aid of the Abel transformation, that

$$\sum_n{}'' = \sum_n{}''\frac{1}{\lambda_n\lambda_{n+1}}\left|\sum_{k=n-\lambda_n+2}^{n+1}(\lambda_n-n-1+k)\frac{\mu_k}{k}kA_k(z)\right| =$$

$$= O(1)\sum_n{}''\frac{1}{\lambda_n^2}\left\{\sum_{k=n-\lambda_n+2}^{n}k|t_k(z)|\left|(\lambda_n-n-1+k)\frac{\mu_k}{k}-(\lambda_n-n+k)\frac{\mu_{k+1}}{k+1}\right|+\right.$$

$$\left.+(n-\lambda_n)|t_{n-\lambda_n+1}(z)|\frac{\mu_{n-\lambda_n+2}}{n-\lambda_n+2}+(n+1)|t_{n+1}(z)|\frac{\lambda_n\mu_{n+1}}{n+1}\right\} \equiv \sum_n{}''_1+\sum_n{}''_2+\sum_n{}''_3.$$

Since $\left|(\lambda_n-n-1+k)\frac{\mu_k}{k}-(\lambda_n-n+k)\frac{\mu_{k+1}}{k+1}\right| \leqq \lambda_k\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)+\frac{\mu_k}{k}$ we have that

$$\sum_n{}''_1 \leqq \sum_{k=2}^{\infty}|t_k(z)|\left(k\lambda_k\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)+\mu_k\right)\sum_{n\geqq k}{}''\frac{1}{\lambda_n^2}.$$

Because $\sum''$ has only the indices n having the property $\lambda_{n+1}>\lambda_n$, it follows that

$$\sum_{n\geqq k}{}''\frac{1}{\lambda_n^2} \leqq \sum_{\nu=\lambda_k}^{\infty}\frac{1}{\nu^2} = O(1)\frac{1}{\lambda_k},$$

hence we obtain, by (2. 1) and (2. 2) that

$$\Sigma_n'' {}_1 = O(1) \sum_{k=2}^{\infty} k|t_k(z)| \left(\frac{\mu_k}{k} - \frac{\mu_{k+1}}{k+1}\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda_k} |t_k(z)| = O(1)$$

for almost all z. Since, by (2. 2) $B(z) = O(1)$ almost everywhere,

$$\Sigma_n'' {}_2 + \Sigma_n'' {}_3 = O(1) \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |t_n(z)| \equiv O(1) B(z) = O(1)$$

for almost all z, i. e.

$$\Sigma_n'' = O(1)$$

for almost all z, too.

This completes the proof of Theorem 1.

Proof of Theorem 2. As in the proof of Theorem 1, we have

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda; z) - V_n(\lambda; z)| =$$

$$= O(1) \left\{ \sum_{k=1}^{\infty} k|t_k(z)| \Delta\left(\frac{\mu_k}{k}\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda_k} |t_k(z)| \right\} \equiv O(1)\big(A(z) + B(z)\big).$$

By CAUCHY's inequality and Lemma 2, we get

$$B(z) \leqq \left\{ \sum_{k=1}^{\infty} \frac{k\mu_k^2}{\lambda_k^2} \right\}^{1/2} \left\{ \sum_{k=1}^{\infty} \frac{|t_k(z)|^2}{k} \right\}^{1/2} = O(1)$$

for almost all z and

$$A(z) \leqq \left\{ \sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 \right\}^{1/2} \left\{ \sum_{k=1}^{\infty} \frac{|t_k(z)|^2}{k} \right\}^{1/2}.$$

In order to prove the theorem, it is sufficient to demonstrate that

$$\sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 < \infty. \tag{2.3}$$

Since

$$\left(\Delta\left(\frac{\mu_n}{n}\right)\right)^2 \leqq 4\left(\frac{1}{n^2}(\Delta\mu_n)^2 + \frac{\mu_n^2}{n^4}\right),$$

so we have that

$$\sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 \leqq 4 \sum_{k=1}^{\infty} k(\Delta\mu_k)^2 + O(1).$$

From this, by Lemma 3, (2. 3) follows, that is, the theorem is proved.

Proof of Theorem 3. Let $V_n(\lambda; x)$ denote the n-th de la Vallée Poussin mean of series (6). Using that

$$A_n(x) = \frac{1}{\pi} \int_0^{\pi} \varphi(t) \cos nt\, dt. \tag{2.4}$$

we obtain the equality

$$d_n(x) \equiv \pi |V_{n+1}(\lambda; x) - V_n(\lambda; x)| =$$

$$= \left| \int_0^\pi \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \cos kt \, dt \right|.$$

We write

$$d_n(x) \leqq d_n^1(x) + d_n^2(x) \equiv \left| \int_0^{1/n} \right| + \left| \int_{1/n}^\pi \right|.$$

By (5), we get that

$$d_n^1(x) = O(1) \left(\frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \int_0^{1/n} |\varphi(t)| \, dt \right) =$$

$$= O(1) \left(\frac{1}{\lambda_n^2 \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \right).$$

Let $\alpha_k^{(n)} = [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k)/k$. For $d_n^2(x)$ with the aid of the Abel transformation, we obtain

$$d_n^2(x) \leqq \left| \int_{1/n}^\pi \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \left\{ \sum_{k=n-\lambda_n+2}^{n} C_k(t) \Delta \alpha_k^{(n)} \right\} dt \right| +$$

$$+ \left| \int_{1/n}^\pi \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \frac{(\lambda_{n+1} - \lambda_n)(1-\lambda_n) + \lambda_n}{n - \lambda_n + 2} \mu(n - \lambda_n + 2) C_{n-\lambda_n+1}(t) \, dt \right| +$$

$$+ \left| \int_{1/n}^\pi \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \frac{\lambda_n \mu(n+1)}{n+1} C_{n+1}(t) \, dt \right| \equiv I_1 + I_2 + I_3.$$

In the following steps we shall use that

$$\int_{1/n}^\pi \frac{|\varphi|}{t} \, dt = O(1); \tag{2.5}$$

in fact, considering (4) and (5), we have

$$\int_{1/n}^\pi \frac{|\varphi|}{t} \, dt = \left(\frac{\Phi}{t} \right)_{1/n}^\pi + \int_{1/n}^\pi \frac{\Phi}{t^2} \, dt = O(1) + \int_{1/n}^\pi \frac{\mu\left(\frac{1}{t}\right)}{t^2 \lambda\left(\frac{1}{t}\right)} \, dt =$$

$$= O(1) + \int_{1/\pi}^n \frac{\mu(x)}{\lambda(x)} \, dx = O(1).$$

By the Lemma 5 and (2. 5), we have

$$I_1 = O\left(\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)}\right),$$

$$I_2 = O\left(\frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+2)\right)$$

and

$$I_3 = O\left(\frac{\mu(n)}{\lambda_n}\right).$$

From the above analysis we obtain that

$$d_n(x) = O\left(\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1} k\alpha_k^{(n)} + \frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} + \right.$$
$$\left. + \frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+2) + \frac{\mu(n)}{\lambda_n}\right).$$

From this it follows

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda; x) - V_n(\lambda; x)| =$$
$$= O\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1} k\alpha_k^{(n)} + \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} + \right.$$
$$\left. + \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+1) + \sum_{n=1}^{\infty}\frac{\mu(n)}{\lambda_n}\right) = O(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4).$$

The fourth sum is finite by (4). Let $\underset{n}{\Sigma'_k}$ and $\underset{n}{\Sigma''_k}$ $(k=1, 2, 3)$ be defined as in Theorem 1. In the estimations of $\underset{n}{\Sigma'_k}$, we shall use that the inside lower indices $(n-\lambda_n+2)$ are strictly increasing. So we have by (4)

$$\underset{n}{\Sigma'_1} = \sum_n{}' \frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1} \mu(k) \leqq \sum_{k=1}^{\infty}\frac{\mu(k)}{\lambda_k} = O(1),$$

$$\underset{n}{\Sigma'_2} = \sum_n{}' \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+2}^{n} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right) \leqq \sum_{k=1}^{\infty} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right) \leqq$$
$$\leqq \sum_{k=1}^{\infty}\left(\Delta\mu(k) + \frac{\mu(k)}{k}\right) = O(1)$$

and

$$\underset{n}{\Sigma'_3} = \sum_n{}' \frac{1}{\lambda_n}\mu(n-\lambda_n+1) \leqq \sum_n{}' \frac{\mu(n-\lambda+1)}{\lambda_{n-\lambda_n+1}} \leqq \sum_{k=1}^{\infty}\frac{\mu(k)}{\lambda_k} = O(1).$$

In the case $\lambda_{n+1}>\lambda_n$, $\alpha_k^{(n)}=(\lambda_n+k-n-1)\dfrac{\mu(k)}{k}$, so we get that

$$\Sigma_{1,n}'' = \sum_n{}'' \frac{1}{\lambda_n^3} \sum_{k=n-\lambda_n+2}^{n+1} (\lambda_n+k-n-1)\mu(k) \leqq$$

$$\leqq \sum_n{}'' \frac{1}{\lambda_n^3} \sum_{k=n-\lambda_n+2}^{n+1} \lambda_k \mu(k) \leqq \sum_{k=2}^{\infty} \lambda_k \mu(k) \sum_{n>k}{}'' \frac{1}{\lambda_n^3}.$$

Because in Σ'' there are only the indices n having the property $\lambda_{n+1}>\lambda_n$, it holds

$$\sum_{n>k}{}'' \frac{1}{\lambda_n^3} \leqq \sum_{\nu=\lambda_k}^{\infty} \frac{1}{\nu^3} = O\left(\frac{1}{\lambda_k^2}\right),$$

so it is easy to see that

$$\Sigma_{1,n}'' = O\left(\sum_{k=2}^{\infty} \frac{\mu(k)}{\lambda_k}\right) = O(1).$$

Abel's transformation gives that

$$\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} \leqq \sum_{k=n-\lambda_n+2}^{n} \alpha_k^{(n)}, \tag{2.6}$$

so we have by $n-\lambda_n \geqq k-\lambda_k$ $(n>k)$

$$\Sigma_{2,n}'' \leqq \sum_n{}'' \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} (\lambda_n+k-n-1)\frac{\mu(k)}{k} \leqq$$

$$\leqq \sum_n{}'' \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \frac{\lambda_k \mu(k)}{k} \leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{n\geqq k}{}'' \frac{1}{\lambda_n^2} \leqq$$

$$\leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{\nu=\lambda_k}^{\infty} \frac{1}{\nu^2} = O\left(\sum_{k=2}^{\infty} \frac{\mu(k)}{k}\right) = O(1).$$

Finally

$$\Sigma_{3,n}'' = \sum_n{}'' \frac{1}{\lambda_n^2}\mu(n-\lambda_n+1) \leqq \mu(1) \sum_n{}'' \frac{1}{\lambda_n^2} \leqq \mu(1) \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = O(1).$$

From the above analysis we obtain the statement of Theorem 3.

Proof of Theorem 4. The proof is similar to the proof of Theorem 3. The first difference steps at (2. 5); that is, under the conditions (7) and (8), we can only say that

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t}\,dt = O(\log\log n).$$

In fact

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t}\,dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2}\,dt = O(1) + \int_{1/n}^{\pi} \frac{1}{t\log\frac{1}{t}}\,dt =$$

$$\leqq O(1) + O(\log\log n) = O(\log\log n).$$

So we have only the following estimations:

$$I_1 = O\left(\frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)}\right),$$

$$I_2 = O\left(\frac{\log\log n}{\lambda_n^2}\left[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n\right]\mu(n-\lambda_n+2)]\right)$$

and

$$I_3 = O\left(\frac{(\log\log n)\mu(n)}{\lambda_n}\right).$$

With the aid of these and the estimations obtained in the proof of Theorem 3, by (7), we obtain that

$$\sum_{n=4}^{\infty} |V_{n+1}(\lambda;x) - V_n(\lambda;x)| = O(1) +$$

$$O\left(\sum_{n=4}^{\infty} \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} + \right.$$

$$\left. + \sum_{n=4}^{\infty} \frac{\log\log n}{\lambda_n^2}\left[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n\right]\mu(n-\lambda_n+1)\right).$$

We can demonstrate the finitness of these sums as in Theorem 3. Let us see e. g. the case of the first sum. Let Σ'_1 and Σ''_1 denote the suitable sums as in Theorem 3.
(7), $\Delta\left(\frac{\mu(k)}{k}\right) \leqq \frac{1}{k}\Delta\mu(k) + \frac{\mu(k)}{k^2}$ and $n-\lambda_n < n+1-\lambda_{n+1}$ give

$$\Sigma'_1 = \sum_n{}' \frac{\log\log n}{\lambda_n} \sum_{k=n-\lambda_n+2}^{n} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right) \leqq$$

$$\leqq 2\sum_{k=2}^{\infty} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right)\log\log k \leqq 2\sum_{k=2}^{\infty} (\log\log k)\Delta\mu(k) + O(1) =$$

$$= O(1)\sum_{k=2}^{\infty}\left(\sum_{n=2}^{k} \frac{1}{n\log n}\right)\Delta\mu(k) + O(1) = O(1)\sum_{k=2}^{\infty} \frac{\mu(k)}{k\log k} < \infty.$$

Using (2. 6), $\alpha_k^{(n)} = (\lambda_n + k - n - 1)\frac{\mu(k)}{k}$ (for $\lambda_{n+1} > \lambda_n$) and $n-\lambda_n \geqq k-\lambda_k$ $(n>k)$, we have

$$\Sigma_1'' = \sum_n{}'' \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} (\lambda_n+k-n-1)\frac{\mu(k)}{k} \leqq$$

$$\leqq \sum_n{}'' \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \frac{\lambda_k\mu(k)}{k} \leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{n\geqq k}^{n-\lambda_n+2\leqq k} \frac{\log\log n}{\lambda_n^2} \leqq$$

$$\leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{\nu=\lambda_k}^{\infty} \frac{\log\log(\nu+k)}{\nu^2} = O\left(\sum_{k=2}^{\infty} \frac{\mu(k)\log\log k}{k}\right) = O(1).$$

We can prove similarly that the second sum is finite, so Theorem 4 follows.

Proof of Theorem 5. If the sequence $\left\{\mu(n)n\Big/\binom{n+\alpha}{n}\right\}$ is monotone, then the statement of Theorem 5 follows from the Corollary with this sequence, namely it is well known that if

$$p_n = \binom{n+\alpha-1}{\alpha-1} \qquad (\alpha>0)$$

then the Nörlund mean reduces to the Cesàro mean of order α.

In the general case we give a short direct proof, but only under conditions (9) we shall detail it because the other case is similar. Using (2. 4), an easy computation gives that for the Cesàro means of series (6)

$$\pi n(\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)) = \int_0^\pi \varphi(t)\left(\frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1}\mu(k)k\cos kt\right)dt \equiv \tau_n^\alpha(x).$$

We write

$$\tau_n^\alpha(x) = \tau_n^\alpha(1;x)+\tau_n^\alpha(2;x) = \int_0^{1/n} + \int_{1/n}^{\pi}.$$

From (9) we get that

$$\tau_n^\alpha(1;x) = O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu\int_0^{1/n}|\varphi(t)|\,dt\right) =$$

$$= O\left(\frac{1}{n^{2\alpha}}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu\right).$$

For $\tau_n^\alpha(2;x)$ with the aid of the Abel transformation, we obtain

$$\pi\tau_n^\alpha(2;x) = \int_{1/n}^{\pi}\varphi(t)\frac{1}{A_n^\alpha}\left\{\sum_{\nu=1}^{n-1}C_\nu(t)(\mu(\nu)A_{n-\nu}^{\alpha-1}-\mu(\nu+1)A_{n-\nu-1}^{\alpha-1})\right\}dt+$$

$$+\int_{1/n}^{\pi}\varphi(t)\frac{1}{A_n^\alpha}\mu(n)C_n(t)\,dt \equiv I_1+I_2.$$

Let $d_\nu = |\mu(\nu)A_{n-\nu}^{\alpha-1} - \mu(\nu+1)A_{n-\nu-1}^{\alpha-1}|$. As (2. 5) is valid now, so by the Lemma 5, we have

$$I_1 = O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n-1} \nu\, d_\nu \int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt\right) = O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n-1} \nu\, d_\nu\right)$$

and

$$I_2 = O\left(\frac{\mu(n) n}{n^\alpha} \int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt\right) = O\left(\frac{\mu(n) n}{n^\alpha}\right).$$

From the above analysis we obtain that

$$\sum_{n=2}^{\infty} |\sigma_n^\alpha(x) - \sigma_{n-1}^\alpha(x)| = O\left(\sum_{n=2}^{\infty} \frac{1}{n^{1+2\alpha}} \sum_{\nu=1}^{n} \mu(\nu)(n+1-\nu)^{\alpha-1}\nu + \right.$$

$$\left. + \sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=1}^{n-1} \nu\, d_\nu + \sum_{n=2}^{\infty} \frac{\mu(n)}{n^\alpha}\right).$$

The third sum is finite by (9). ABEL's transformation gives that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1+2\alpha}} \sum_{\nu=1}^{n} \mu(\nu)(n+1-\nu)^{\alpha-1}\nu \leqq \sum_{\nu=1}^{\infty} \mu(\nu)\nu \sum_{n=\nu}^{\infty} (n+1-\nu)^{\alpha-1} \frac{1}{n^{1+2\alpha}} =$$

$$= O(1) \sum_{\nu=1}^{\infty} \mu(\nu)\nu \frac{1}{\nu^{1+\alpha}} = O(1),$$

i. e. the first sum is finite, too. Putting $\bar{n} = \left[\frac{n}{2}\right]$, we can write the second sum into two sums:

$$\sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=1}^{n} \nu\, d_\nu = \sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=1}^{\bar{n}} \nu\, d_\nu + \sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=\bar{n}+1}^{n-1} \nu\, d_\nu. \tag{2. 7}$$

The first sum under (2. 7) is less than

$$\sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=1}^{\bar{n}} \nu(\mu(\nu)A_{n-\nu}^{\alpha-1} - \mu(\nu+1)A_{n-\nu}^{\alpha-1}) + O(1) =$$

$$= O(1) \sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=1}^{\bar{n}} \nu n^{\alpha-1} \Delta\mu(\nu) = O(1) \sum_{\nu=1}^{\infty} \nu\, \Delta\mu(\nu) \sum_{n=2\nu}^{\infty} \frac{1}{n^2} =$$

$$= O(1) \sum_{\nu=1}^{\infty} \Delta\mu(\nu) = O(1).$$

Similarly, the second sum under (2. 7) is not greater than

$$O(1) \sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha}} \sum_{\nu=\bar{n}+1}^{n-1} \nu(n-\nu)^{\alpha-1} \Delta\mu(\nu) + O(1) = O(1) \sum_{\nu=2}^{\infty} \Delta\mu(\nu) = O(1).$$

We have also that

$$\sum_{n=2}^{\infty} |\sigma_n^\alpha(x) - \sigma_{n-1}^\alpha(x)|$$

converges, so our statement is proved.

Proof of Theorem 6. It runs similarly to the proof of Theorems 3 and 4.

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